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Supercritical Behavior of an Ordered Trajectory

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The behavior of a particle undergoing a discrete, stepped motion around a circle is investigated for the case in which the average number of steps per revolution is the golden mean. The new features of this study are that the map in question is not invertible and that consequently the orbit is bunched into intervals within the circle. Because of this bunching, one can calculate the motion in some detail.

KEY WORDS: Circle map; ordered trajectory; scaling; universal.

1. INTRODUCTION

A mapping problem is one in which one investigates the properties of a sequence of points $z_0, z_1, z_2, \ldots,$ each point being generated from the last by the application of a defined function R:

$$z_{j+1} = R(z_j) \tag{1.1}$$

These problems serve as simple models of dynamical behavior in which one can think of the z_j as a description of the state of the system at a time $t_j = j\tau$. One is particularly interested in universal or generic properties of the set z_j—that is, properties which do not depend in detail upon the form of R. Any such robust property has a chance of being important for the behavior of real, and more complex, dynamical systems.

One set of universal behavior recently investigated[1-3] concerns maps in which z is simply a real number and R is a member of a family of maps $R_\Omega(z)$ which obey a kind of periodicity condition

$$R_\Omega(z+1) = 1 + R_\Omega(z) \tag{1.2a}$$

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In this case one can think of motion on a circle of unit circumference and interpret z as a description of the total distance moved on that circle. The Ω dependence of the family is simply linear:

$$R_\Omega(z) = R_0(z) + \Omega \tag{1.2b}$$

Additional requirements include the statement that $R_0(z)$ be smooth, i.e., at least continuous. Four examples of such maps treated in this paper are

$$R_0(z) = z - \frac{k}{2\pi}\sin 2\pi z \tag{1.3a}$$

which was studied in Refs. 1–4, and also

$$R_0(z) = \begin{cases} (z/\lambda)^2 & \text{for} \quad 0 \leqslant z \leqslant \lambda \\ 1 & \text{for} \quad \lambda \leqslant z \leqslant 1 \end{cases} \tag{1.3b}$$

and also

$$R_0(z) = \begin{cases} 16[z/\lambda(1 - z/\lambda)]^2 & \text{for} \quad 0 \leqslant z \leqslant \lambda/2 \\ 1 & \text{for} \quad \lambda/2 \leqslant z < 1 \end{cases} \tag{1.3c}$$

and finally

$$R_0(z) = \begin{cases} z/\lambda & \text{for} \quad 0 \leqslant z \leqslant \lambda \\ 1 & \text{for} \quad \lambda \leqslant z \leqslant 1 \end{cases} \tag{1.3d}$$

In the latter three cases $R_0(z)$ has flat regions, i.e., regions in which $R_0'(z) = 0$.

The trajectories produced by all these and many other maps are characterized by a winding number

$$\omega(\Omega, z_0) = \lim_{j\to\infty} \frac{z_j}{j} \tag{1.4}$$

which describes the average number of revolutions traveled per step. As indicated, the winding number is a function of parameters in R_Ω (e.g., Ω) and also of the initial point in the trajectory, z_0. When ω is rational, one can describe the motion as phase locked or commensurate. One particular case of commensurate motion is a cyclical motion in which $z_{j+q} = z_j + p$ with p and q being integers. Then, if q is greater than zero and is the smallest integer which makes this condition possible, we have a cycle of length q in which the particle undergoes p revolutions per cycle. Clearly $\omega = p/q$ in this case.

It is also possible to have motions with irrational winding numbers. These incommensurate motions are called KAM trajectories and they are particularly interesting because they serve as borderline cases. Very near by in k and Ω or in z_j there exist orbits with very different behaviors. For small k, $|k| < 1$, these nearby orbits can be very long cycles. For $|k| > 1$, the nearby orbits can be long cycles or they can be much more chaotic trajectories. It is possible that the cases of incommensurate motion can be used to organize and understand the other motions.

One winding number has been subjected to particular study, the one in which $\omega = (\sqrt{5} - 1)/2$, that is the inverse of the golden mean. We shall use the symbol W to represent this particular value of ω, and we shall get our main results for this particular winding number.

In our work we shall be particularly interested in the "supercritical domain" that is in $R_0(z)$ having derivatives $R_0'(z)$ which are both positive and negative. In the example of Eq. (1.3a), this case arises when $|k| > 1$. However, this situation is very complex. Instead of treating this case in full detail we focus upon a particularly simple subclass of trajectories: the so-called "ordered orbits." An orbit z_j for $j = 0, \pm 1, \pm 2, \ldots$ is said to be ordered with winding number ω if the orbit points z_j fall onto a circle with unit circumference in exactly the same order as the points $z_j^0 = \omega j$. In symbols this ordering condition is the statement that for all integers j, k, and n

$$\frac{z_j - z_k - n}{(j - k)\omega - n} > 0 \tag{1.5}$$

whenever the denominator in Eq. (1.5) is non-zero. The ordered orbits will be the subject of this paper.

To understand the behavior in the supercritical domain, we shall make heavy use of results which apply to the (simpler) domain in which $R_0(z)$ is a monotonically increasing function of z as in the examples of Eqs. (1.3b–d) and also in the example of Eq. (1.3a) for $|k| < 1$. In this monotonic domain, it is well-known that[7] the winding number ω is independent of z_0,

$$\omega(z_0, \Omega) = \omega(\Omega) \tag{1.6}$$

and is a continuous and monotonically nondecreasing function of Ω. In particular, we shall study features of the $\omega = W$ orbits of models (1.3b–d) which also apply to the correspondingly ordered supercritical orbits of model (1.3a).

The subcritical domain is characterized by the condition that $R_0(z)$ always has slope greater than zero. The critical domain has $R_0'(z) \geqslant 0$, with some points of zero slope. These two conditions are attained by model (1.3a), respectively, for $|k| < 1$ and $k = 1$. The other models (1.3b–d) are always critical. There is one additional condition satisfied by the subcritical orbits with irrational winding number which is not satisfied in the supercritical or critical domains. Define orbit points on the circle by

$$x_j = z_j - n_j \tag{1.7}$$

where n_j, the integer, is chosen to make x_j lie in the interval $[0, 1)$. For almost any irrational winding number, including $\omega = W$, any x in the interval $[0, 1)$ will be an accumulation point of the x_j. Furthermore, the density of orbit points $\rho(x)$ will be a smooth function of x which remains nonzero for all x.

At criticality, this smoothness disappears.[1-3] There are an infinite number of points for which $\rho(x)$ vanishes. In the supercritical domain the behavior of an orbit with winding number W becomes even more singular. In the next section of this paper, we shall give a set of arguments which show that surely an orbit indeed exists. This argument shows in addition that the orbit points cannot be spread upon the whole circle but that instead they are bunched into regions. A more and more precise look will define the regions more and more narrowly until one sees that the orbit points be in a set of disconnected regions of total measure zero within the circle. This is the opposite of the subcritical case in which the orbit filled whole circle with finite density.

In Section 3, a simplified model—which holds in the large-k region—is derived by looking at the ordered trajectories which arise from the model (1.3a). Since these trajectories are bunched into narrow regions not all of the z domain is relevant for their treatment. For the most stable, and most interesting trajectories we are concerned with the regions of $R(z)$ near its extrema. We thus argue for a model of the form (1.3b) with $\lambda \ll 1$. This approach is acceptable because no orbit point falls in the flat region of $R(z)$. In this third section, we also calculate in low order in λ some properties of the ordered irrational-ω trajectory.

Section 4 is devoted to the derivation of the asymptotic-large j—properties of this trajectory. By using the bunching properties, one can predict quite accurately where each point in the trajectory will fall. This permits a kind of scaling or renormalization group analysis of the behavior of z_j for large j, starting from $z_0 = 0$. One of the key results is that for $j = F_{2m}$, where F_k is a Fibonacci number,

$$z_{F_{2m}} = F_{2m-1} + D 2^m \exp\left[-A_+ \alpha_+^m - A_+ \alpha_-^m \right] \tag{1.8}$$

Here A_+, A_-, and D are nonuniversal constants (which, for example, depend on λ) but

$$\alpha_\pm = 1 \pm \sqrt{2}/2 \tag{1.9}$$

are universal critical indices. Equation (1.8) is supposed to be universally accurate in the limit of large m.

In this section, we also look at another mapping problem corresponding to the model (1.3c) in which there are two extrema in $R_0(z)$ in a unit interval of z and in which the infinitely long trajectory passes arbitrarily close to both extrema. The behavior of the cycle elements are very different in this situation. Instead of Eq. (1.8) one has

$$z_{F_{2m}} = F_{2m-1} + D \exp\left[-\frac{m^2}{3} - A_+ m - A_-(\alpha_-)^m \right] \tag{1.10}$$

with $\alpha_- = 1/4$ and A_+, A_-, and D being nonuniversal and λ-dependent constants.

2. DEFINITION AND DESCRIPTION OF SUPERCRITICAL ORDERED TRAJECTORIES

This chapter is concerned with giving an outline description of the qualitative properties of ordered trajectories, most specifically supercritical ordered trajectories with winding number, $W = (\sqrt{5} - 1)/2$. Section 2.1 describes the behavior in the monotone domain and gives proof of several of the observations made in Chapter 1. Section 2.2 describes the particular ordering properties of golden mean trajectories, i.e., those with $\omega = W$. In the last section, results are extended to the supercritical case.

2.1. The Monotonic Domain

In the monotonic domain $R_\Omega(z)$ satisfies, in addition to Eqs. (1.2), the condition that if $z > z'$

$$R_\Omega(z) \geqslant R_\Omega(z') \tag{2.1}$$

It is very easy to show that the winding number is independent of z. Define the result of J iterations of R_Ω upon z_0 as $z_J(\Omega, z_0)$. Because of the periodicity $z_J(\Omega, z_0 + 1) = 1 + z_J(\Omega, z_0)$. Now pick z to obey $z_0 < z < z_0 + 1$. Because of the monotonicity

$$z_J(\Omega, z_0) \leqslant z_J(\Omega, z) \leqslant 1 + z_J(\Omega, z_0)$$

From the definition (1.4), then

$$\omega(\Omega, z_0) \leqslant \omega(\Omega, z) \leqslant \omega(\Omega, z_0)$$

Thus the winding number is independent of z, in this monotonic case.

An equally simple and well-known argument shows the ordering of trajectories. Let $\omega(\Omega)$ be the winding number of R_Ω. Define in analogy to Eq. (1.5)

$$\Gamma_{P,Q}(z_0) = \frac{z_Q(\Omega, z_0) - P - z_0}{Q\omega(\Omega) - P} \tag{2.2}$$

We wish to prove that if $\omega(\Omega) \neq P/Q$, then it is always true that $\Gamma_{P,Q}(z_0)$ is greater than zero. First, this quantity can never be zero. If it were zero then there would be a cycle with $\omega = P/Q \neq \omega(\Omega)$. But this violates the condition that there is only one winding number. If for a given P, Q, $\Gamma_{P,Q}(z)$ were negative for any z, it would be negative for all z. Since it is periodic in z and continuous there exists $\epsilon_{P,Q} > 0$ such that for all z

$$\Gamma_{P,Q}(z) < -\epsilon_{P,Q} \tag{2.3}$$

Then for any positive integer N, Eqs. (2.2) and (2.3) imply

$$\Gamma_{NP,NQ}(z_0) = \frac{z_{NQ}(\Omega, z_0) - NP - z_0}{NQ\omega(\Omega) - NP} \leqslant -\epsilon_{P,Q} < 0 \tag{2.4}$$

As $N \to \infty$, the definition of the winding number implies that $\Gamma_{NP,NQ}(z_0) \to 1$. Thence we get the contradiction $1 < 0$. Thence (2.3) must be wrong. Therefore $\Gamma_{P,Q}(z_0)$ is always positive under the conditions stated and Eq. (1.5) is true. As a consequence, for any orbit the fractional parts x_j and x_k of z_j and z_k are ordered in exactly the same way as the fractional parts of $j\omega(\Omega)$ and $k\omega(\Omega)$.

By a variant of the same argument, one can prove that in the monotonic domain, if $\omega(\Omega)$ is rational and equal to p/q—with p and q being relatively prime integers—then there exists a cycle $x_0, x_1, \ldots, x_q = x_0 + p$ with winding number $\omega(\Omega)$ and length q. To achieve this result notice that

$$\gamma_{p,q}(z_0) = z_q(\Omega, z_0) - z_0 - p$$

must change sign. If it did not and were say always greater than $\epsilon > 0$ then $\omega(\Omega) > (p/q + \epsilon)$—which is false.

The monotonic increase of $\omega(\Omega)$ with Ω is a consequence of the montonicity of $R_\Omega(z)$ in Ω and z.

Next argue for the continuity of $\omega(\Omega)$. For each Q and Ω there will exist an integer, $P(Q,\Omega)$, such that for all z_0

$$P(Q,\Omega) - 1 < z_Q(z_0, \Omega) - z_0 < P(Q,\Omega) + 1$$

Further by continuity there exists an interval $(\Omega_-(P, Q), \Omega_+(P, Q))$ (including Ω) such that when Ω' lies in this interval

$$P(Q,\Omega) - 1 < z_Q(z, \Omega') - z < P(Q,\Omega) + 1$$

Thus, from Eq. (1.5)

$$\frac{P(Q,\Omega) - 1}{Q} < \omega(\Omega') < \frac{P(Q,\Omega) + 1}{Q} \tag{2.5}$$

As $Q \to \infty$, Eq. (2.5) implies the continuity of $\omega(\Omega')$.

Finally, continuity plus the statement $\omega(\Omega + 1) = \omega(\Omega) + 1$ implies that $\omega(\Omega)$ takes on all possible values. Thus, for example, there exists an Ω value, Ω^*, such that $\omega(\Omega^*) = W$.

2.2. Golden Mean Ordering

The ordering just described turns out to be particularly simple in the "golden mean" case in which $\omega = W$. Let the elements of the orbit be $z_j = x_j + n_j$, where $z_0 = 0$, n_j is an integer and x_j is the fractional part of z_j. In the monotonic domain, these orbit elements x_j will be ordered in exactly the same way as $wj - n_j$ for all integer values of j.

To discuss the ordering of the latter quantities use the Fibonacci numbering system,[8] which is based upon the Fibonacci numbers, F_n, defined by

$$F_{n+1} = F_n + F_{n-1}, \qquad F_0 = F_1 = 1 \tag{2.6}$$

We shall represent any positive integer J by writing it as a sum of nonadjacent Fibonacci numbers, i.e., as

$$J = \sum_{n=1}^{\infty} F_n \sigma_n(J) \tag{2.7a}$$

where $\sigma_n(J)$ is either zero or one and

$$\sigma_n(J)\sigma_{n+1}(J) = 0 \tag{2.7b}$$

All positive integers can be represented in this form, and the representation is unique.[8] This representation is useful because, for large n, WF_n is very close to the integer F_{n-1}.

Specifically

$$WF_n = F_{n-1} - (-W)^{n+1} \tag{2.8}$$

so that Eq. (2.7a) permits splitting WJ into an integer and a remainder according to

$$WJ = \sum_{n=1}^{\infty} F_{n-1}\sigma_n(J) - \sum_{n=1}^{\infty} (-W)^{n+1}\sigma_n(J) \tag{2.9}$$

One more definition is necessary. The class of J, $C(J)$ is defined as the lowest value of n for which $\sigma_n(J)$ is nonzero. Table I gives the decomposition of the first few integers and their class value. Note that for even classes the remainder term in Eq. (2.9) is positive, while for odd $C(J)$ it is negative. Hence one can write the remainder term in Eq. (2.9) in terms of the fractional part of WJ, as

$$\{WJ\} = (1/2)\left[1 + (-1)^{C(J)+1}\right] - (-W)^{C(J)+1} - \sum_{n=C(J)+2}^{\infty} (-W)^{n+1}\sigma_n(J) \tag{2.10}$$

For positive J, the x_J for the orbit are ordered in exactly the same manner

Table I. Decomposition of Integers into Sums of Fibonacci Numbers. The Class is Defined as the First Value of k for Which $\sigma_k(J)$ is Nonzero.

	$\sigma_k(J)$ for k =					
Integer, J	1	2	3	4	5	Class
1	1					1
2	0	1				2
3	0	0	1			3
4	1	0	1			1
5	0	0	0	1		4
6	1	0	0	1		1
7	0	1	0	1		2
8	0	0	0	0	1	5
9	1	0	0	0	1	1
10	0	1	0	0	1	2
11	0	0	1	0	1	3
12	1	0	1	0	1	1

as $\{WJ\}$. Figure 1 shows how this ordering works out for the first few integers.

Notice how $C(J)$ serves as the primary determinant of the ordering. Elements of different classes fall in disjoint regions with even-class regions falling on the left and off-class regions on the right. If $C(J) = 2m$, higher m values appear further to the left. For odd values of $C(J)$, higher classes appear further to the right. In particular if $C(J) = 2m$, then the fractional part, according to Eq. (2.10), lies in the interval

$$\{-WF_{2m+1}\} < \{WJ\} < \{-WF_{2m-1}\} \tag{2.11a}$$

while if $C(J) = 2m + 1$

$$\{-WF_{2m}\} < \{WJ\} < \{-WF_{2m+2}\} \tag{2.11b}$$

[The derivation of Eq. (2.11a) depends upon the statements $\{-WF_{2m+1}\} = W^{m+2}$ and also $W^2 + W = 1$.]

Corresponding to the statements (2.11) about the ordering of $\{WJ\}$, we have exact statements about the ordering of the orbit elements x_j for

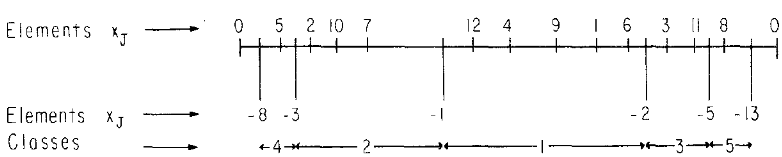


Fig. 1. Arrangements of trajectory elements within the unit interval for an ordered trajectory with $\omega = W$. The first two lines list J values for the elements x_j. The third line shows class intervals.

$J \geqslant 0$. We see that x_J will lie in class $2m$ if

$$x_{-F_{2m+1}} < x_J < x_{-F_{2m-1}} \tag{2.12a}$$

While it falls in class $2m+1$ if, for $m > 1$,

$$x_{-F_{2m}} < x_J < x_{-F_{2m+2}} \tag{2.12b}$$

Figure 1 includes these negative-J orbit elements to show how they serve as class boundaries.

2.3. Supercritical Behavior of Ordered Trajectories

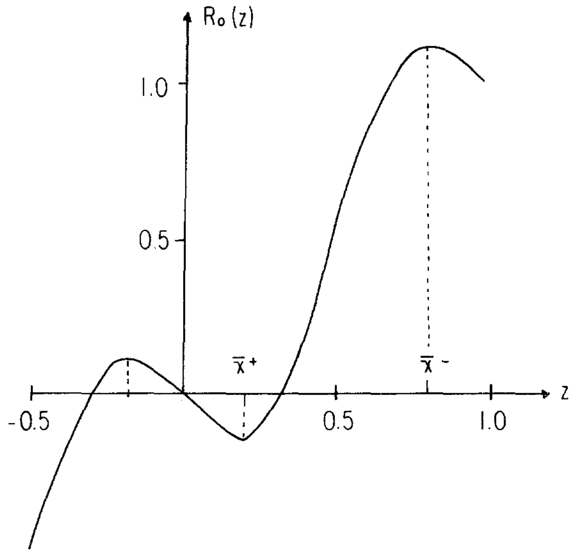
In the supercritical domain [e.g., for model (1.3a) with $k > 1$] $R'(z)$ can be negative. In general, in this case the orbits are not ordered and they are very complex. When trajectory elements pass through a region in which $R'(z) < 0$ they can get out of order. However, even in the supercritical case, there exist ordered orbits with all possible winding numbers, including rational values and also such irrational values as $\omega = W$.

To see how these orbits arise, start from $R_0(z)$ as shown in Fig. 2a. The problem with this $R_0(z)$ is that it is supercritical, i.e., it has a region of negative slope. One can "cure" this by inserting a flat region as in Figs. 2b, 2c, or 2d. Figure 2b is the general case. The resulting function has a flat region but no region of slope approaching zero outside the flat region. We call this function $R_0^F(z)$. If the flat region just touches the point of zero slope and lies to the left of it as in Fig. 2c, we call the resulting function $R_0^+(z)$, while we call the flattening of Fig. 2d, in which the zero slope appears to the left of the flat region, $R_0^-(z)$. These figures plot $R_0^F(z)$ for $-1/2 < z < 1$. To get the rest of the function we demand $R_0^F(z+1) = R_0^F(z) + 1$.

Imagine that we found a trajectory for $R_0^F(z)$. We can use the ideas which apply to the monotonic case in finding this trajectory. If it just so happened that this trajectory had no elements inside the flat regions then the trajectory would equally well apply to $R_0(z)$. We would have found our required supercritical trajectory.

But since $R_0^F + \Omega$ is a monotonic map, we know that there is an Ω value for which the winding number is W. Set Ω equal to this value. Then consider the trajectory generated by $R_0^F + \Omega$, with the starting point x_0. The subsequent trajectory elements $z_j = x_j + n_j$, $j = 1, 2, \ldots$ must have the x_j properly ordered (i.e., in the same way as $\{Wj\}$) in the interval $[0, 1)$. In fact no x_j can fall into the forbidden regions $[0, x_0)$ or $[x_0', 1)$ since if this happened there would be a cycle of length $j-1$, and hence a winding number different from W.

Note also that for each $z \neq x_1$ (mod 1) there exists a unique preimage z' with $R_0^R(z') + \Omega = z$. Thus one can step by step construct z_j and x_j for



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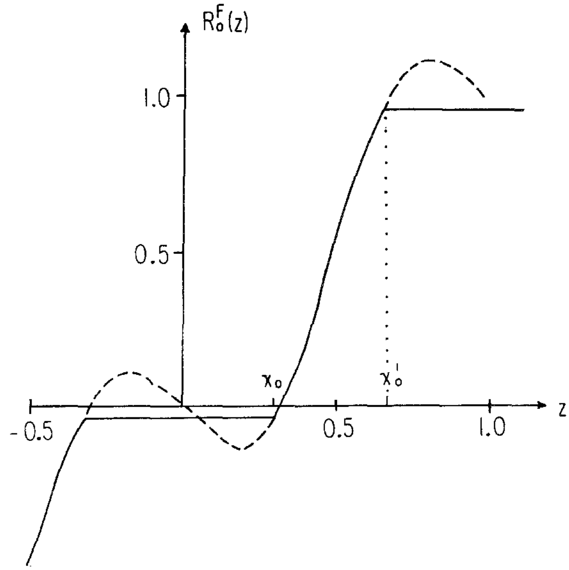

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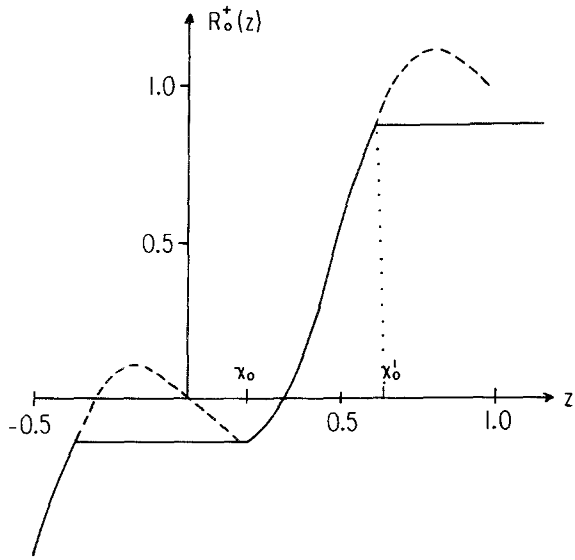

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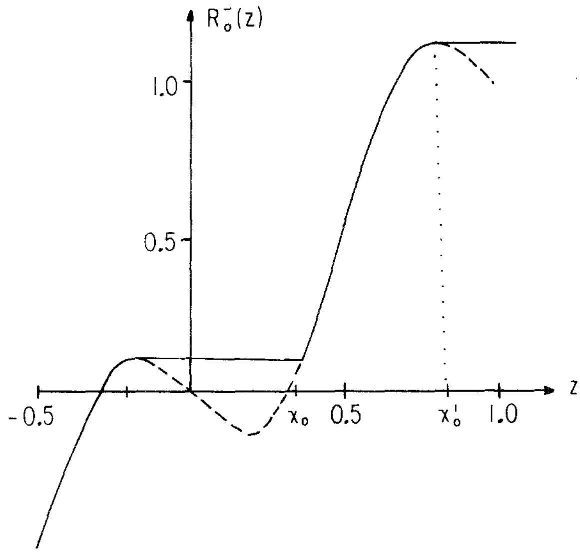

d

Fig. 2. Flattening of the supercritical map. Figure 2a shows the starting map which has extrema at $z = \bar{x}^{\pm} + n$, where n is an integer. In 2b, 2c, 2d flattened versions of the map are shown with the limits of the flat regions being $x_0' + n - 1$ and $x_0 + n$. In 2c and 2d the quadratic extrema appear at the edges of the increasing regions.

negative values of j. No negative-j x_j can equal x_0, since then there would be a cycle of length $(-j+1)$. And equally well these elements cannot be in the forbidden, flat, regions of R_0^F. In this way we have constructed a trajectory z_j with winding number W which is equally well an ordered trajectory for $R_0^F + \Omega$ and for the supercritical map $R_0 + \Omega$.

By exactly the same logic one can construct another ordered trajectory z_j' with this same winding number by starting out with the element $z_0' = x_0'$. Let x_j' be the fractional part of z_j'. By construction $z_1' = z_1 + 1$ and hence for j positive $z_j' = z_j + 1$ and $x_j' = x_j$. But since $x_0' \neq x_0$, for negative j the elements of these two cycles are different.

We know the relative orderings within each of the trajectories x_j and x_k'. To get the relative orderings of the terms in the different cycles, notice that the map $R_0^F + \Omega$ takes the interval $[x'_{-1}, x_{-1}]$ into the forbidden interval $[x_0', x_0 + 1]$, while j applications of the map (for $j > 0$) take the interval $[x'_{-j}, x_{-j}]$ into a version of the forbidden interval displaced by an integer. It is always true for $J > 0$ that $x'_{-j} < x_{-j}$. No trajectory elements can fall into the intervals $[x'_{-j}, x_{-j}]$ since, if they did there would be a finite length cycle. Hence these intervals are all forbidden to any trajectory element. For this reason, these intervals must be nonoverlapping.

Thus we know everything about the relative ordering of the two trajectories x_j and x_j'. In particular,

(a) all elements fall into $[x_0, x_0']$;

(b) for $j > 0$, $x_j = x_j'$;

(c) for $j < 0$, $x_j' < x_j$;

(d) if $j < 0$, $[x_j', x_j]$ is a forbidden region;

(e) if $j \neq k$ the relative ordering of x_j and x_k' is of the same as the ordering of $\{Wj\}$ relative to $\{Wk\}$. Figure 3 depicts the ordering of cycle elements given by these rules. Note once again how classes separate.

Last, we should point out that x_0 and x_0' are each accumulation points for trajectory elements. Assume that they were not. Then there would be some accumulation points z_k^*, $k = 0, \pm 1, \pm 2$ which would themselves form

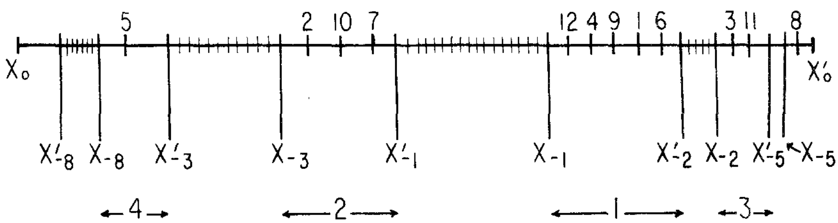

Fig. 3. Arrangement of trajectory elements within the interval $[x_0, x_0']$ for an ordered trajectory with $\omega = W$. The top line shows j values for x_j. The bottom line shows class intervals. The hatched regions are forbidden intervals.

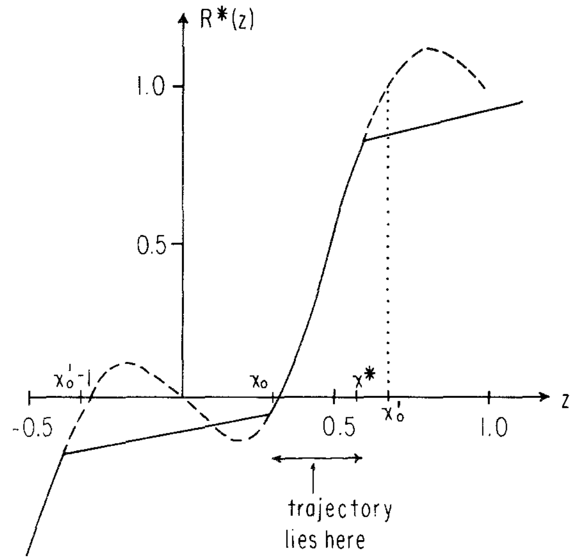

Fig. 4. A subcritical map. If the accumulation points of trajectory elements lie in the interval $[x_0, x^*]$ with $x_0 < x^* < x_0'$, then this trajectory can be derived from a subcritical map. If $\omega = W$, this is impossible.

a trajectory with winding number W if $R_0(z) + \Omega$. If these accumulation points were bounded away from, say x_0', by having a maximum x^* at $x^* < x_0'$ this trajectory would be an ordered trajectory of the map $R^*(x) + \Omega$, where $R^*(x)$ as depicted in Figure 4, would be subcritical. But every subcritical map has an $\omega = W$ trajectory which accumulates over the entire unit interval of x. Since by construction no elements fall in $[x'_{-1}, x_{-1}]$ we have reached a contradiction. Both x_0 and x_0' must be accumulation points of trajectory elements as must be all other trajectory elements.

2.4. Types of Ordered Trajectories

Notice that we have in essence proved that a given supercritical $R(z) = R_0(z) + \Omega$ has a whole family of ordered trajectories. Figure 2b depicts a family of $R_0^F(z)$ which depends on where the horizontal line is placed. The two limiting cases are given by Figs. 2c and 2d. In these cases the trajectories pass through x_0 or x_0', which are extrema of $R(z)$. In the intermediate cases the trajectories avoid the extrema. One should not be surprised if the limiting cases lie in a different universality class from the intermediate cases.

As Ω is varied, one would expect that there is a whole interval (Ω_-^*, Ω_+^*) in which there exists a supercritical ordered trajectory with winding number W, with the ends of the interval corresponding to trajectories which avoid the flat regions of Figs 2c and 2d, respectively.

We shall be most interested in the extreme motions in which the trajectory passes through one extremum since, in some sense, we expect this motion to be the more stable.

Finally, in special cases, which we shall not further describe here, we might have a trajectory with the very same ordering as described here but for which x_0 and x_0' are both extrema of $R_0(x)$. Model (1.3c) has this property. This situation defines yet another kind of ordered supercritical trajectory.

3. A MODEL FOR SUPERCRITICAL BEHAVIOR

3.1. Simplified Models

For $k > 1$, the map of Eq. (1.3)

$$R_\Omega(z) = z + \Omega - \frac{k}{2\pi}\sin 2\pi z \tag{3.1}$$

is supercritical in that $R_\Omega'(0) < 0$. Therefore intervals around $z = 0$, $\pm 1, \pm 2, \ldots$ become forbidden territory so no elements of an ordered orbit may appear in them. As k increases, these forbidden regions become wider and wider until for very large k only small intervals about $z = \pm 1/4$, $\pm 3/4, \ldots$ become accessible. Near $z = 1/4$, Eq. (3.1) becomes

$$R_\Omega(z) \approx z + \Omega - \frac{k}{2\pi} + \pi k(z - 1/4)^2 \tag{3.2}$$

Now we wish to obtain the one-extremum map of Eq. (1.3b). An extremum appears in (3.2) at

$$\bar{z} = 1/4 - (2\pi k)^{-1} \tag{3.3}$$

We then rewrite the mapping problem in terms of a new variable $z - \bar{z}$, and redefine Ω according to

$$\Omega + \frac{k}{2\pi} - \frac{1}{4\pi k} \to \Omega$$

Thus in the neighborhood of these extrema, we have a new map defined for $z \approx n$, with n integral

$$R_\Omega(z) = \Omega + n + \left(\frac{z-n}{\lambda}\right)^2 \tag{3.4}$$

Here, λ is related to the original k via

$$\lambda^2 = (\pi k)^{-1}$$

In the region of large k for ordered orbits passing near the extremum the map (3.4) is essentially equivalent to (3.1), and simpler.

The next step of analysis is to force the ordered behavior by putting flat regions into $R(z)$ as indicated in Fig. 2b. Consider the case depicted in Fig. 2c in which x_0 appears at the minimum if $R(z)$. From Eq. (3.4) x_0 is zero. The other parameter used in the flattening is x_0', defined via

$$R_\Omega(x_0') = R_\Omega(x_0) + 1 \tag{3.5}$$

According to Eq. (3.4) $x_0' = \lambda$.

Thus the flattened map becomes equivalent to Eq. (1.3b), i.e.,

$$R_0^+(z) = \begin{cases} n + \left(\dfrac{z-n}{\lambda}\right)^2 & \text{for} \quad n \leqslant z \leqslant n + \lambda \\ n+1 & \text{for} \quad n + \lambda \leqslant z \leqslant n+1 \end{cases} \tag{3.6}$$

We wish to understand the trajectories of (3.6) for small λ for winding number equal to W.

An equivalent analysis can be carried out for trajectories which stay away from the extrema of Eq. (3.1). Let the trajectories have x values which fall between x_0 and x_0'. We have $(1/4 < x_0 < 3/4)$ and x_0' just greater than x_0. Let $k \gg 1$. Then for z near x_0 Eq. (3.1) implies

$$R_\Omega(z) = \Omega - \frac{k}{2\pi}\sin 2\pi x_0 + x_0 + (1 - k\cos 2\pi x_0)(z - x_0)$$

Now choose

$$\lambda = (1 - k\cos 2\pi x_0)^{-1} \tag{3.7}$$

and pick k to be large enough so that $0 < \lambda \ll 1$. If we replace $\Omega - (k/2\pi)\sin 2\pi x_0$ by Ω and shift variables according to $z - x_0 \to z$, we have a map which looks like

$$R(z) = \frac{z}{\lambda} + \Omega \qquad \text{for} \quad |z| \ll 1 \tag{3.8}$$

Our periodicity and flattening requirements then give the particularly simple map of Eq. (1.3d):

$$R_0^F(z) = \begin{cases} \dfrac{z-n}{\lambda} + n & \text{for} \quad n \leqslant z \leqslant n + \lambda \\ n+1 & \text{for} \quad n + \lambda \leqslant z \leqslant n+1 \end{cases} \tag{3.9}$$

We can analyze Eq. (3.9) to get an ordered orbit for $k \gg 1$ which actually arises from the supercritical map (3.1).

3.2. Golden Mean Trajectory: Case 1: No Extrema

The quasilinear map of Eq. (3.9) produces particularly simple ordered orbits with winding number W. The map is, in the interesting region,

$$R_\Omega(z) = \Omega + n + \frac{z-n}{\lambda} \qquad \text{for} \quad 0 \leqslant z - n \leqslant \lambda$$

and has an inverse

$$R_\Omega^{-1}(z) = \lambda(z - \Omega - n) + n \qquad \text{for} \quad 0 < (z - \Omega - n) < 1 \tag{3.10}$$

The first few orbit points are then $x_0 = 0$, $x'_0 = \lambda$, and $x_1 = \Omega$. Since the ordering implies $x_0 < x_1 < x'_0$ we must have the parameter Ω lying in the interval $0 < \Omega < \lambda$.

A more useful evaluation of Ω is given by listing additional orbit points in order of increasing x value as

$$\begin{aligned} x_0 &= z_0 = 0 \\ x_2 &= z_2 - 1 = \Omega + \Omega/\lambda - 1 \\ x'_{-1} &= z'_{-1} + 1 = \lambda(\lambda - \Omega) \\ x_{-1} &= z_{-1} + 1 = \lambda(1 - \Omega) \\ x_1 &= z_1 = \Omega \\ x'_0 &= z'_0 + 1 = \lambda \end{aligned} \tag{3.11}$$

Since $x_0 < x_2 < x'_{-1}$, we find

$$\Omega = \lambda - \lambda^2 + \lambda^3 + \cdots \tag{3.12}$$

where $\cdots$ indicates terms of higher order in λ. At this point, we have the following approximate evaluation of orbit elements

$$\begin{aligned} x_0 &= 0 \\ x_2 &= 0x\lambda^2 + \cdots \\ x'_{-1} &= \lambda^3 - \lambda^4 + \cdots \\ x_{-1} &= \lambda - \lambda^2 + \lambda^3 - \lambda^4 + \cdots \\ x_1 &= \lambda - \lambda^2 + \lambda^3 + \cdots \\ x'_0 &= \lambda \end{aligned} \tag{3.13}$$

To get higher-order results, we need to go two steps further and calculate $x_3 = z_3 - 1 = x_2/\lambda + \Omega - 1$ and $x_4 = z_4 - 2 = R(x_3) - 1$. Since we know $x_{-1} < x_4 < x_1$ we then find that

$$x_1 = \Omega = \lambda - \lambda^2 + \lambda^3 - \lambda^5 + \lambda^6 + \cdots \tag{3.14}$$

From the evaluation (3.14) and the expressions for the orbit elements, we can calculate their power series expansions as

$$x_J = \sum_k C_k \lambda^k$$

The data obtained in this way are listed in Table II.

Table II. Expansion of Trajectory Elements in a Power Series in λ for the Quasilinear Map. The Expansion is $x = \sum_k C_k \lambda^k$.

x	Class	C_1	C_2	C_3	C_4	C_5	C_6	C_7
x_0	∞	0	0	0	0	0	0	0
x'_{-3}	4	0	0	0	0	0	0	0
x_{-3}	2	0	0	1	−1	0	0	0
x_2	2	0	0	1	−1	0	—	—
x'_{-1}	2	0	0	1	−1	0	1	1
x_{-1}	1	1	−1	1	−1	0	1	−1
x_4	1	1	−1	1	—	—	—	—
x_1	1	1	−1	1	0	−1	1	1
x'_{-2}	1	1	−1	1	0	−1	1	1
x_{-2}	3	1	0	0	0	−1	1	0
x_3	3	1	0	0	0	—	—	—
x'_{-5}	3	1	0	0	0	−1	1	0
x_5	5	1	0	0	0	0	0	0
x'_0	∞	1	0	0	0	0	0	0

In our further analysis, we shall pay considerable attention to the behavior of the x's in a given class. For even classes, we define a typical class element as

$$x^{(2m)} = x_{F_{2m}} - x_0 \tag{3.15a}$$

and the class width by

$$\Delta x_{2m} = x'_{-F_{2m-1}} - x_{-F_{2m+1}} \tag{3.15b}$$

The corresponding class element for odd m is measured from the other end point

$$y^{(2m+1)} = x'_0 - x_{F_{2m+1}} \tag{3.15c}$$

while the class width is

$$\Delta y_{2m+1} = x'_{-F_{2m+2}} - x_{-F_{2m}} \tag{3.15d}$$

Notice that as $m \to \infty$ the class elements get closer and closer to the end points. For example, we see that

$$y^{(1)} \approx \lambda^2, \qquad y^{(3)} \approx \lambda^5 \tag{3.16a}$$

while $y^{(5)}$ must be of order equal to or higher than λ^8. Similarly

$$x^{(2)} \approx \lambda^3 \tag{3.16b}$$

while $x^{(4)}$ must be smaller or equal to order λ^8.

In each case, the widths of the classes appears to be much smaller than the distance to the end points. Thus, for example, we have

$$\begin{aligned} \Delta y_1 &\approx \lambda^4 \\ \Delta y_3 &\lesssim O(\lambda^8) \\ \Delta x_2 &\approx \lambda^6 \end{aligned} \tag{3.16c}$$

Because these widths are so small, we shall be able to handle all elements of a class as a unit.

3.3. Golden Mean Trajectory: Case 2: One Extremum

For this case, we can rewrite Eq. (3.6) as an expression for x, $x_{j+1} = R_\Omega^+(x_j)$, with

$$R_\Omega^+(x) = \begin{cases} \Omega + (x/\lambda)^2 & \text{for} \quad 0 < x < x'_{-1} \\ \Omega - 1 + (x/\lambda)^2 & \text{for} \quad x_{-1} < x < \lambda \end{cases} \tag{3.17}$$

The inverse of this map is

$$G_\Omega^+(x) = \begin{cases} \lambda(x - \Omega + 1)^{1/2} & \text{for} \quad 0 < x < \Omega \\ \lambda(x - \Omega)^{1/2} & \text{for} \quad \Omega < x < \lambda \end{cases} \tag{3.18}$$

The first few cycle elements are given by

$$\begin{aligned} x_0 &= 0 \\ x'_{-1} &= \lambda(\lambda - \Omega)^{1/2} \\ x_{-1} &= \lambda(1 - \Omega)^{1/2} \\ x_1 &= \Omega \\ x'_0 &= \lambda \end{aligned} \tag{3.19}$$

That the elements are ordered in the same way as in the listing (3.19) implies that as $\lambda \to 0$, $\Omega \approx \lambda$. The next order is given by calculating

$$\begin{aligned} x_2 &= (\Omega/\lambda)^2 + \Omega - 1 \\ x'_{-2} &= \lambda\left[1 - \Omega + \lambda(\lambda - \Omega)^{1/2}\right]^{1/2} \\ x_{-2} &= \lambda\left[1 - \Omega + \lambda(1 - \Omega)^{1/2}\right]^{1/2} \end{aligned}$$

Thus, for example, the statement $x_{-1} < x_1 < x'_{-2}$ gives

$$\Omega = \lambda - \lambda^2/2 + O(\lambda^3)$$

To obtain the next-order result quickly notice that $x_{-2} = \lambda - O(\lambda^4)$ and hence since $x_3 > x_{-2}$, $x_3 - \lambda$ must have an order which is higher than or equal to λ^4. Thus

$$x_2 = G_\Omega^+(x_3) = \lambda(\lambda - \Omega)^{1/2} + O(\lambda^4) = \Omega - 1 + (\Omega/\lambda)^2$$

Finally, then, we can solve this equation to find

$$\Omega = -\lambda^2/2 + \left[(\lambda^4/4) + \lambda^2 + \lambda^3(\lambda - \Omega)^{1/2}\right]^{1/2} + O(\lambda^5)$$

which then gives

$$\begin{aligned} \Omega &= \lambda - \lambda^2/2 + \gamma\lambda^3 - \gamma\lambda^4/2\sqrt{2} \\ \gamma &= 1/8 + \sqrt{2}/4 \end{aligned} \tag{3.20}$$

Working from Eq. (3.20), we get a set of lowest order results for the x's and y's, in particular

$$\begin{aligned} y^{(1)} &= \lambda^2/2 \\ \Delta y_1 &= \lambda^3/2\sqrt{2} \\ x^{(2)} &= \lambda^2/\sqrt{2} \\ \Delta x_2 &= \lambda^4/8 \end{aligned} \tag{3.21}$$

in the limit as $\lambda \to 0$. Once more, we see that the widths of the class intervals are much smaller than the distance to the nearest end point.

4. RECURSIVE ANALYSES

4.1. The Recursion Equations

The mode of analyses carried out in the last two sections is much more difficult than it need be. If we simply assume that all x_j's in a given class are tightly bunched together, we can obtain a much simpler and more powerful way of computing class properties. Simply assert that if x_j lies in class c

$$b_c = \frac{d}{dx} R(x)\bigg|_{x = x_j} \tag{4.1}$$

is, to a good approximation, independent of which particular element is being considered. As a corollary consider

$$B_c = \frac{d}{dx} (R)^{F_c - 1}(x)\bigg|_{x = x_j} \tag{4.2}$$

when x_J lies in class c. All such x_J's have, during the next $F_c - 1$ steps of iteration essentially the same "class history." That is all these iterates fall into the same class in the same sequence. In fact they fall into class "1" $F_c - 2$ times, into class "l" F_{c-l-1} times, and into class "$c-1$" once. Equation (4.1) then converts into the statement

$$B_c = \prod_{l=1}^{c-1} (b_l)^{F_{c-l-1}} \tag{4.3}$$

In turn, Eq. (4.3) implies a recursion relation for B_c, namely,

$$\begin{aligned} B_{c+1} &= B_c b_c B_{c-1} \\ B_0 &= B_1 = 1 \end{aligned} \tag{4.4}$$

Equation (4.4) is one of the major inputs of our recursive analysis. To get the other crucial part consider the quantities

$$Q_{2m} = (R)^{F_{2m}}\left(x_0 + x^{(2m)}\right) - (R)^{F_{2m}}(x_0) \tag{4.5a}$$

$$Q_{2m+1} = (R)^{F_{2m+1}}(x_0') - R^{F_{2m+1}}\left(x_0' - y^{(2m+1)}\right) \tag{4.5b}$$

On one hand, we can evaluate

$$Q_{2m} = x_{2F_{2m}} - x_{F_{2m}} \tag{4.6}$$

Since $2F_{2m} = F_{2(m-1)} + F_{2m+1}$, it follows that x_{2F_m} is in class $m-1$ for $m > 1$ and class 1 for $m = 1$. Since $x^{(c-2)} \gg x^{(c)}$ and since the classes are tightly bunched we have

$$Q_{2m} \approx \begin{cases} x^{(2m-2)} & \text{for} \quad m > 1 \\ x_1 - x_0 & \text{for} \quad m = 1 \end{cases} \tag{4.7a}$$

A similar analysis for Q_{2m+1} gives

$$Q_{2m+1} = \begin{cases} y^{(2m-1)} & \text{for} \quad m \geqslant 1 \\ x_1 - x_0 & \text{for} \quad m = 1 \end{cases} \tag{4.7b}$$

On the other hand, we can also write

$$Q_{2m} = (R)^{F_{2m}-1}\left[R\left(x_0 + x^{(2m)}\right)\right] - (R)^{F_{2m}-1}\left[R(x_0)\right]$$

Since $x^{(2m)}$ is small the quantity

$$\epsilon_{2m} = R\left(x_0 + x^{(2m)}\right) - R(x_0) \tag{4.8a}$$

is very small, as is the corresponding odd-class quantity

$$\epsilon_{2m+1} = R(x_0') - R\left[x_0' - y^{(2m+1)}\right] \tag{4.8b}$$

Then, if we expand Q to first order in ϵ we find $Q_c = \epsilon_c B_c$. When we apply

Eq. (4.7) we then get our crucial result

$$\epsilon_c B_c = \begin{cases} x^{(c-2)} & \text{if } c \text{ is even and } > 2 \\ y^{(c-2)} & \text{if } c \text{ is odd and } \geqslant 3 \\ x_1 - x_0 & \text{if } c = 2 \text{ or } 3 \end{cases} \tag{4.9}$$

Equations (4.4), (4.8), and (4.9) form a complete set of relations which can be solved to obtain, for example, $x^{(2m)}$.

Corresponding analysis can be applied to obtain recursion relations for $\Delta x_{2m} = x'_{-F_{2m-1}} - x_{-F_{2m+1}}$ and $\Delta y_{2m+1} = x'_{-F_{2m+2}} - x_{-F_{2m}}$. By taking F_{2m-1} recursions on each of the trajectory elements in Δx_{2m}, and assuming that Δx_{2m} is small, we find

$$\Delta x_{2m} b_{2m} B_{2m} = x'_0 - x_{-F_{2m}}$$

Since $x_{-F_{2m}}$ forms the boundary of class $2m + 1$, we conclude

$$\Delta x_{2m} b_{2m} B_{2m} = y^{(2m+1)} \tag{4.10}$$

and also, by a similar logic

$$\Delta y_{2m+1} b_{2m+1} B_{2m} = x^{(2m+2)} \tag{4.11}$$

All the equations we have derived apply whenever the class elements are sufficiently tightly bunched. If $R(z)$ is very smooth except for singularities near the boundaries, a sufficient condition is that

$$\begin{aligned} \Delta y_{2m+1}/y^{(2m+1)} &\ll 1 \\ \Delta x_{2m}/x^{(2m)} &\ll 1 \end{aligned} \tag{4.12}$$

These conditions hold for our model problems for small values of m, at least when λ is small. As we shall see, they hold generically for large m. Hence, the recursion relation of this section holds generically for large m.

Thus, from these relations, we shall obtain generically true statements about supercritical ordered trajectories.

4.2. The Quasilinear Model

The recursive analysis of the previous section can be trivially carried through to answers if the map has the form given in Eq. (3.9). In this quasilinear case $R'(z) = 1/\lambda$ for all z's lying in the trajectory. Hence b_c of Eq. (4.2) is independent of c and is given by

$$b_c = b = 1/\lambda \tag{4.13}$$

correspondingly Eq. (4.2) gives

$$B_c = b^{F_c - 1} \tag{4.14}$$

which automatically obeys Eq. (4.4). Equations (4.8) imply

$$\epsilon_{2m} = bx^{(2m)}$$

$$\epsilon_{2m+1} = by^{(2m+1)}$$

Then Eqs. (4.9) imply, for example, that

$$x^{(2m)}/x^{(2m-2)} = b^{-F_{2m}}$$

$$x^{(2)}(b)^2 = \lambda$$

which then directly imply that

$$x^{(2m)} = \lambda^{F_{2m+1}} \tag{4.15a}$$

Similar logic can be employed to obtain

$$y^{(2m+1)} = \lambda^{F_{2m+2}} \tag{4.15b}$$

Equations (4.10) and (4.11) directly give

$$\Delta x_{2m} = \left[x^{(2m)}\right]^2 \tag{4.15c}$$

$$\Delta y_{2m+1} = \left[y^{(2m+1)}\right]^2 \tag{4.15d}$$

All these results directly check against the low-order results of Eqs. (3.16) and (3.17). This agreement supports the correctness of the logic given here.

Notice also that the conditions (4.12) are certainly satisfied either for λ small or for general λ with m large.

4.3. Generic Case—No Extrema

For the case in which the trajectory passes through no extrema, the generic situation is hardly more complex than the quasilinear model discussed above. Require that as c goes to infinity the cycle elements approach the end points and that

$$b_c \to \begin{cases} b & \text{if } c \text{ is even} \\ b' & \text{if } c \text{ is odd} \end{cases}$$

The large class solution to Eqs. (4.4) are readily given as

$$\begin{aligned} \ln B_{2m} &= \ln D_B + A_+ W^{-2m} + A_-(-W)^{2m} \\ \ln B_{2m+1} &= \ln D_B' + A_+ W^{-(2m+1)} + A_-(-W)^{2m+1} \end{aligned} \tag{4.16}$$

Here A_+ and A_-, D_B and D_B' are nonuniversal, but W is, of course, a universal characterizer of the trajectory. For the analysis to be correct we must have $A_+ > 0$.

Since ϵ_{2m} is proportional to $x^{(2m)}$ one finds immediately the generic answers

$$\ln x^{(2m)} = \ln D - A_+ W^{-(2m+1)} - A_-(-W)^{2m+1} \tag{4.17a}$$

and also that

$$\ln y^{(2m+1)} = \ln D' - A_+ W^{-(2m+2)} - A_-(-W)^{2m+2} \tag{4.17b}$$

4.4. One Extremum

A less trivial case arises when there is one quadratic extremum, at $x = x_0$. Then as $c \to \infty$, we find

$$b_c \to \begin{cases} b & \text{if } c \text{ is odd} \\ \dfrac{2x^{(c)}}{\lambda^2} & \text{if } c \text{ is even} \end{cases} \tag{4.18}$$

correspondingly,

$$\epsilon_c \to \begin{cases} by^{(c)} & \text{if } c \text{ is odd} \\ \left[\dfrac{x^{(c)}}{\lambda}\right]^2 & \text{if } c \text{ is even} \end{cases} \tag{4.19}$$

Of course, for the quadratic model these results are true even for small c. In that case, $b = 2/\lambda$. From Eqs. (4.9) we find that

$$bB_{2m+1}y^{(2m+1)} = y^{(2m-1)} \tag{4.20}$$

and also

$$b_{2m}^2 B_{2m} = 2b_{2m-2} \tag{4.21}$$

Generically, these last equations apply for large m. In the quadratic model they hold down to $m = 1$ and 2, respectively, and they are supplemented by the "boundary conditions"

$$bB_1 y^{(1)} = \lambda \tag{4.22a}$$

$$b_2^2 B_2 = 2b \tag{4.22b}$$

Finally, write the recursion relations (4.4) for B as

$$B_{2m+1} = b_{2m}B_{2m}B_{2m-1} \tag{4.23}$$

$$B_{2m+2} = bB_{2m+1}B_{2m} \tag{4.24}$$

which then is supplemented by the boundary conditions $B_0 = B_1 = 1$.

Low-order results for the quadratic model are easily recovered from these equations. From Eqs. (4.23) and (4.24), we find that $B_2, B_3, \ldots$ are given by b, bb_2, b^3b_2, $b^4b_2^2b_4$, $b^8b_2^3b_4$, and finally $B_7 = b^{14}b_2^5b_4^2b_6$. From Eq. (4.22b) and (4.21), we then find successively that to lowest order in λ

$$\begin{aligned} b_2 &= \sqrt{2}\,, & x^{(2)} &= \lambda^2/\sqrt{2} \\ b_4 &= b^{-3/2}\sqrt{2}\,, & x^{(4)} &= \lambda^{7/2}/4 \\ b_6 &= b^{-4}2^{-1/4}, & x^{(6)} &= \lambda^6/2^{5+1/4} \end{aligned} \tag{4.25a}$$

In addition, Eqs. (4.22a) and (4.20) give

$$\begin{aligned} y^{(1)} &= \lambda^2/2 \\ y^{(3)} &= \lambda^4/2^{7/2} \\ y^{(5)} &= \lambda^{15/2}/2^{17/2} \end{aligned} \tag{4.25b}$$

It is also relatively easy to solve the recursion equations. By putting together Eqs. (4.23) and (4.21), we find

$$b_{2m}B_{2m+1} = 2b_{2m-2}B_{2m-1}$$

Hence

$$b_{2m}B_{2m+1} = D\,2^{m-1}/b \tag{4.26}$$

where D is a constant to be determined. Using Eqs. (4.21), (4.24), and (4.26) we can eliminate all B's from our analysis. First, we find a recursion equation for b_{2m} alone, namely,

$$\frac{b_{2m}^4}{(b_{2m+2})^2 b_{2m-2}} = 2^{m-1}D \tag{4.27}$$

Then, Eq. (4.20) yields

$$B_{2m+2}y^{(2m+1)} = y^{(2m-1)}B_{2m}$$

or

$$y^{(2m+1)} = 2C/B_{2m+2} \tag{4.28}$$

where C is a constant to be determined. From (4.21), we then evaluate

$$y^{(2m+1)} = Cb_{2m+2}^2/b_{2m} \tag{4.29}$$

Next, go after Δx_{2m}, the width of the class $2m$. From Eqs. (4.10) and (4.18)

$$\frac{\Delta x_{2m}}{x^{(2m)}} = \frac{2y^{(2m+1)}}{\lambda^2 b_{2m}^2 B_{2m-1}}$$

This expression reduces to

$$\frac{\Delta x_{2m}}{x^{(2m)}} = \frac{Cb}{\lambda^2 2^{2m-4} D^2} b_{2m} \tag{4.30}$$

Since b_{2m} goes to zero as $m \to \infty$ we can be sure that our class intervals really do get sufficiently narrow as $m \to \infty$. Hence our analysis is certain to be correct for large m.

For the quadratic model evaluate D by setting $m = 1$ in Eq. (4.26). Then

$$D = 8/\lambda^2 \tag{4.31a}$$

Find C by setting $m = 0$ in Eq. (4.28). Then

$$C = \lambda/2 \tag{4.31b}$$

Equation (4.30) then states that

$$\frac{\Delta x_{2m}}{x^{(2m)}} = \frac{\lambda^2 b_{2m}}{2^{2m+2}} \tag{4.32}$$

so that $\Delta x_{2m}/x_{2m} \ll 1$ for small λ for all m in the quadratic model. For $m = 1$, we can check Eq. (4.32) against Eq. (3.21) and find that we have made no errors.

The general solution of Eq. (4.28) is very easy. Guess a result of the form

$$b_{2m} = D2^m \exp\left[-A_+(\alpha_+)^m - A_-(\alpha_-)^m\right] \tag{4.33}$$

Notice that this solution works if

$$\alpha_\pm^2 - 2\alpha_\pm + 1/2 = 0$$

or

$$\alpha_\pm = 1 \pm \sqrt{2}/2 \tag{4.34}$$

Then just so long as $A_+ > 0$, we have a valid solution. Here $\alpha_\pm$ are, of course, universal while D, A_+, and A_- depend upon the details of the mapping. From Eq. (4.18) we find that for large m

$$\ln x^{(2m)} = m \ln 2 - A_+(\alpha_+)^m - A_-(\alpha_-)^m + D_x \tag{4.35}$$

where D_x is nonuniversal. Correspondingly from Eq. (4.30) we find

$$\ln y^{(2m+1)} = m \ln 2 - A_+(2\alpha_+ - 1)(\alpha_+)^m - A_-(2\alpha_- - 1)(\alpha_-)^m + D_y \tag{4.36}$$

where D_y is also nonuniversal. Equations (4.35) and (4.36) provide the generic large-m solution of the problem of ordered supercritical trajectories with winding number W when the trajectory passes through one quadratic extremum.

4.5. Both End Points Quadratic

Finally look at a situation in which $R'(x)$ is positive between x_0 and x'_0 but goes linearly to zero at both end points. Then for large m we have the analog of Eqs. (3.18):

$$\begin{aligned} B_{2m+1} &= b_{2m} B_{2m} B_{2m-1} \\ B_{2m+2} &= b_{2m+1} B_{2m+1} B_{2m} \end{aligned} \tag{4.37}$$

Here b_{2m} is proportional to $x_{F_{2m}} - x_0 = x^{(2m)}$ and b_{2m+1} is proportional to $y^{(2m+1)} = x'_0 - x_{F_{2m+1}}$. Since Eqs. (4.37) do not distinguish between even or odd values of c, we can write

$$B_{c+1} = b_c B_c B_{c-1} \tag{4.38}$$

One can go through the same derivation which led to Eq. (4.21) and find for the present case

$$b_c^2 B_c = 2b_{c-2} \tag{4.39}$$

Equations (4.38) and (4.39) combine to give

$$b_c B_{c+1} = 2b_{c-2} B_{c-1}$$

which have the solution

$$\frac{b_c B_{c+1}}{2^{(c+1)/2}} = \begin{cases} D & \text{for } c \text{ even} \\ D' & \text{for } c \text{ odd} \end{cases} \tag{4.40}$$

For simplicity, we specialize to the case in which $D = D'$. (One can always achieve this result by applying a nonlinear coordinate transformation.) Equations (4.40) and (4.39) combine to give a recursion equation for b_c alone, namely,

$$\frac{b_c^2 2^{c/2}}{b_{c-1} b_{c-2}} = D^{-1} \tag{4.41}$$

It is then quite easy to solve Eq. (4.41). Try a solution of the form

$$b_c = F \exp\left[-\gamma c^2 - Ec - A(\alpha)^c \right]$$

This solution will work if

$$2\alpha^2 - \alpha - 1 = 0$$

The root $\alpha = 1$ is not relevant. Take the other root

$$\alpha = -1/2 \tag{4.42}$$

The quantity γ is determined from the recursion equation to obey

$$-2\gamma c^2 + \gamma(c-1)^2 + \gamma(c-2)^2 = -\frac{c}{2}\ln 2 + \text{const}$$

Then $\gamma = (1/12) \times \ln 2$. The solution is finally

$$b_c = F 2^{-c^2/12} \exp(-Ec - A\alpha^c) \tag{4.43}$$

Here F, E, and A are all nonuniversal.

Once again, we would like b_c to go to zero as c becomes infinite. For large c, the behavior of b_c is dominated by the factor of $2^{-c^2/12}$. Thus, b_c always does go to zero as required.

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